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Genus one correction to Seiberg-Witten prepotential from β -deformed matrix model

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ABSTRACT: We study β -deformed matrix models with Penner type potentials, which correspond to $\mathcal{N} = 2$ $SU(2)$ supersymmetric gauge theories with $N_F = 2, 3$, and 4 flavors. We compute explicitly the genus one corrections to the free energy of the matrix model and show that they match the corresponding results obtained from the Nekrasov partition function.

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1 Introduction

The past several years have seen much progress in the study of $\mathcal{N} = 2$ superconformal field theories (SCFTs). It was shown that the compactification of M5-branes on a Riemann surface with punctures gives rise to a class of $\mathcal{N} = 2$ SCFTs [1]. Matter hypermultiplets of these SCFTs are related to the punctures on the Riemann surface. Elementary S-duality transformations are related to different sewings of the same Riemann surface.

Moreover, it was proposed by Alday, Gaiotto and Tachikawa (AGT) [2] that the Nekrasov partition function of a class of $\mathcal{N}=2$ $SU(2)$ quiver SCFTs is identified with a chiral half of the correlation function of Liouville field theory on the corresponding Riemann surfaces. Specifically, the perturbative part of the partition function corresponds to the Liouville three-point function of primary operators and the instanton part of the partition function coincides with the conformal block generated by the Virasoro algebra.

In [3], the AGT relation was generalized to $\mathcal{N} = 2$ SCFTs with gauge group $SU(N_c)$, relating them to the A_{N_c-1} conformal Toda theory. [4] Also, the instanton partition function of pure super Yang-Mills, which is the non-conformal limit of SCFTs obtained by

sending the mass of the matter hypermultiplets to infinity, was identified with the norm of a coherent state, called Whittaker state, in Liouville field theory. [5], [6] Numerous works have been done investigating various aspects of this surprising relation.

In particular, Dijkgraaf and Vafa (DV) [7] observed that the classical spectral curve of a Penner type matrix model can be interpreted as the Seiberg-Witten curve [8], [9] of $\mathcal{N} = 2$ SCFT. They geometrically engineered the gauge theory with a suitable local Calabi-Yau geometry and employed the large N duality of the B-model topological string to describe the gauge theory with the matrix model. Given the CFT description of the matrix model, where the number of insertion of screening operators in Toda theory is the rank N of the matrices, [10], [11], it was suggested that the matrix model bridges between the gauge theory and the Liouville/Toda CFT. Matrix models corresponding to gauge theories with less number of flavors are also proposed in [13]. These matrix models in the context of the AGT relation have attracted much interest. [16]-[31]

According to the DV proposal, the coupling g_s and the Ω background parameter $\epsilon = \epsilon_1 + \epsilon_2 = g_s(\sqrt{\beta} - 1/\sqrt{\beta})$ are of the same order in g_s . Thus, we double expand the free energy in g_s and ϵ , fixing the ratio $Q = \epsilon/g_s$ or equivalently the central charge of Liouville theory. [12] In other words, we consider the free energy as

$$\begin{aligned} F &= \sum_{k,l \geq 0} g_s^{2k} \epsilon^l F_{k,l} \\ &= F_{0,0} + \epsilon F_{0,1} + \epsilon^2 F_{0,2} + g_s^2 F_{1,0} + \cdots \end{aligned} \quad (1.1)$$

In this paper, we will be computing the genus one part $F_{0,2}$ of the free energy in the β -deformed matrix model. It has been shown in [14] that the genus zero free energy of the matrix model agrees with that of the $SU(2)$ gauge theory. This evidence was extended to the half genus case of the β -deformed matrix model in [15]. The genus one correction in the ordinary matrix model, i.e. $\beta = 1$, was studied in [19].

In section 2, we give a quick review of the β -deformed matrix model with logarithmic potentials and derive the spectral curves from the loop equation. In the following three sections, we explicitly calculate the genus one part of the free energy of the matrix models for $N_F = 2, 3$, and 4. In section 6, we summarize and discuss the results. In Appendix A, we present the detailed computation of integrals used in obtaining the free energy for $N_F = 3$ and 4 cases. In appendix B, the Nekrasov instanton partition function is reviewed and the free energy is calculated from them.

2 β -deformed Matrix Model and Spectral Curve

2.1 Penner type Matrix Model

The partition function of β -deformed matrix model is given by

$$Z = \int \left[\prod_{I=1}^N d\lambda_I \right] \Delta^{2\beta} \exp \left[\frac{\sqrt{\beta}}{g_s} \sum_{I=1}^N V(\lambda_I) \right], \quad (2.1)$$

where $\Delta = \prod_{I < J} (\lambda_I - \lambda_J)$ is the Vandermonde determinant. The Ω background parameters are related to the matrix model parameters through

$$\epsilon_1 = g_s \sqrt{\beta}, \quad \epsilon_2 = -\frac{g_s}{\sqrt{\beta}}. \quad (2.2)$$

The potentials for $N_F = 2, 3, 4$ cases are given by

$$V(z)_{N_F=2} = (2\mu_3 + \epsilon) \log z + \Lambda_2 \left(z + \frac{1}{z} \right), \quad (2.3)$$

$$V(z)_{N_F=3} = (2\mu_3 + \epsilon) \log z + 2m_1 \log(z-1) - \frac{\Lambda_3}{z}, \quad (2.4)$$

$$V(z)_{N_F=4} = (2m_0 + \epsilon) \log z + 2m_1 \log(z-1) + 2m_2 \log(z-q), \quad (2.5)$$

where the mass parameters m_0, m_1, m_2 with additional m_∞ , are related to the four anti-fundamental hypermultiplet masses μ_i of the gauge theory by

$$\mu_1 = m_1 + m_\infty, \quad \mu_2 = m_1 - m_\infty, \quad \mu_3 = m_2 + m_0, \quad \mu_4 = m_2 - m_0 \quad (2.6)$$

The Λ_2 and Λ_3 are dimensionful parameters and correspond to the dynamical scales of the gauge theory, whereas q is dimensionless and is identified with the exponential of the UV coupling of the SCFT. The potential for $N_F = 3$ case is derived from $N_F = 4$ potential by taking $\mu_4 \rightarrow \infty$ with fixed $\Lambda_3 \equiv q\mu_4$. The potential for $N_F = 2$ is obtained from the $N_F = 3$ potential by sending $\mu_2 \rightarrow \infty$ while $\Lambda_2^2 \equiv \Lambda_3\mu_2$ is fixed. The neutrality condition in Liouville theory states that

$$\mu_1 + \mu_3 + g_s \sqrt{\beta} N = 0. \quad (2.7)$$

The free energy of the matrix model is defined to be

$$F \equiv g_s^2 \log Z = -\epsilon_1 \epsilon_2 \log Z, \quad (2.8)$$

and it can be expanded in g_s and $\epsilon = \epsilon_1 + \epsilon_2$,

$$F = \sum_{k,l \geq 0} g_s^{2k} \epsilon^l F_{k,l}. \quad (2.9)$$

In ordinary matrix model, where $\beta = 1$, the parameter ϵ vanishes so only those terms with $l = 0$ survive. In this paper, we compute the leading ϵ^2 -correction $F_{0,2}$ to the free energy.

2.2 Spectral Curves from Loop Equation

We define n -point connected resolvent by

$$W(z_1, \dots, z_n) = \beta \left(\frac{g_s}{\sqrt{\beta}} \right)^{2-n} \left\langle \sum_{I_1} \frac{1}{z_1 - \lambda_{I_1}} \cdots \sum_{I_n} \frac{1}{z_n - \lambda_{I_n}} \right\rangle_c. \quad (2.10)$$

The loop equation, which is obtained from the variation of the partition function (2.1) by $\delta\lambda_I = \frac{\alpha}{\lambda_I - z}$ for small α , is given by [15], [32],

$$g_s^2 W(z, z) + W(z)^2 + \epsilon W'(z) + V'(z)W(z) - f(z) = 0, \quad (2.11)$$

where the function $f(z)$ is

$$f(z) = g_s \sqrt{\beta} \sum_I \left\langle \frac{V'(z) - V'(\lambda_I)}{z - \lambda_I} \right\rangle. \quad (2.12)$$

We expand the resolvents and the potential as follows.

$$W(z_1, \dots, z_n) = \sum_{k,l \geq 0} g_s^k \epsilon^l W_{k,l}(z_1, \dots, z_n)$$

$$V(z) = V_0(z) + \epsilon V_1(z)$$

Then, the loop equation (2.11) is expanded to

$$\begin{aligned} g_s^0 \epsilon^0 : & \quad W_{0,0}^2(z) + V_0'(z) W_{0,0}(z) - f(z) = 0 \\ g_s^0 \epsilon^1 : & \quad 2 y_{0,0}(z) W_{0,1}(z) + W_{0,0}'(z) + V_1'(z) W_{0,0}(z) = 0 \\ g_s^0 \epsilon^2 : & \quad 2 y_{0,0}(z) W_{0,2}(z) + W_{0,1}^2(z) + W_{0,1}'(z) + V_1'(z) W_{0,1}(z) = 0 \end{aligned} \quad (2.13)$$

up to ϵ^2 , where we have defined

$$y_{0,0}(z) \equiv W_{0,0}(z) + \frac{1}{2} V_0'(z) = \frac{1}{2} \sqrt{V_0'(z)^2 + 4f(z)}, \quad (2.14)$$

which is the leading order term in the double expansion of the spectral curve,

$$y(z) \equiv W(z) + \frac{1}{2} V'(z) = \sum_{k,l \geq 0} g_s^k \epsilon^l y_{k,l}(z). \quad (2.15)$$

It is easy to see from the expansion of the loop equation (2.11) that $W_{1,0}(z)$ and $W_{1,1}(z)$ vanish.

The filling fraction in the matrix model is identified with the Coulomb branch parameter a of the gauge theory and is given by the integral of the one form $y(z)dz$ along the A-cycle of the spectral curve,

$$\begin{aligned} a &= \frac{1}{2\pi i} \oint_A y(z) dz, \\ &= \sum_{k,l \geq 0} g_s^k \epsilon^l a_{k,l}, \end{aligned} \quad (2.16)$$

where we have double expanded the vev. To compute the period integral, we first express $y_{0,1}(z)$ and $y_{0,2}(z)$ in terms of $y_{0,0}(z) \equiv y_0(z)$ and the potential. From (2.13) and (2.14),

we obtain

$$\begin{aligned} y_{0,1}(z) &= W_{0,1}(z) + \frac{1}{2}V_1'(z) \\ &= -\frac{y_0'}{y_0} + \frac{V_0'' + V_0'V_1'}{2y_0}, \end{aligned} \quad (2.17)$$

$$\begin{aligned} y_{0,2}(z) &= W_{0,2}(z) \\ &= -\frac{1}{2y_0} \left(W_{0,1}^2(z) + W_{0,1}'(z) + V_1'(z)W_{0,1}(z) \right) \\ &= -\frac{1}{2y_0} \left(y_{0,1}^2 + y_{0,1}' - \frac{1}{4}V_1'^2 - \frac{1}{2}V_1'' \right). \end{aligned} \quad (2.18)$$

Using the expression (2.17) of $y_{0,1}(z)$, we write $y_{0,2}(z)$ as

$$y_{0,2}(z) = \frac{y_0'^2}{8y_0^3} - \frac{(V_0'' + V_0'V_1')^2}{32y_0^3} + \frac{V_1'^2 + 2V_1''}{8y_0} + \frac{d}{dz} \left(\frac{2y_0' - V_0'' - V_0'V_1'}{8y_0^2} \right) \quad (2.19)$$

Thus, the subleading terms of the vector multiplet vev read

$$a_{0,1} = \frac{1}{2\pi i} \oint \frac{V_0'' + V_0'V_1'}{2y_0} dz, \quad (2.20)$$

$$a_{0,2} = \frac{1}{2\pi i} \oint \left(\frac{y_0'^2}{8y_0^3} - \frac{(V_0'' + V_0'V_1')^2}{32y_0^3} + \frac{V_1'^2 + 2V_1''}{8y_0} \right) dz, \quad (2.21)$$

where we have dropped from $a_{0,1}$ the shift caused by the total derivative of $\log y_0$. [15]

3 $N_F = 2$ Model

We first consider the matrix model which is dual to the $SU(2)$ gauge theory with $N_F = 2$. The action is given by

$$V(z) = V_0(z) + \epsilon V_1(z) = 2\mu_3 \log z + \Lambda \left(z + \frac{1}{z} \right) + \epsilon \log z, \quad (3.1)$$

where we have omitted the subscript from Λ_2 in (2.3). The function $f(z)$ in (2.12) is evaluated to be

$$f(z) = \frac{c_1}{z} + \frac{c_2}{z^2}, \quad (3.2)$$

with

$$c_1 = g_s \sqrt{\beta} \left\langle \sum_{I=1}^N \left(-\frac{2\mu_3 + \epsilon}{\lambda_I} + \frac{\Lambda}{\lambda_I^2} \right) \right\rangle = g_s \sqrt{\beta} N \Lambda = -(\mu_1 + \mu_3) \Lambda, \quad (3.3)$$

$$c_2 = g_s \sqrt{\beta} \Lambda \left\langle \sum_{I=1}^N \frac{1}{\lambda_I} \right\rangle, \quad (3.4)$$

where the equation of motion $\langle \sum_I V'(\lambda_I) \rangle = 0$ and (2.7) have been used in computing c_1 . The planar spectral curve (2.14) becomes

$$y_0(z)^2 = \frac{\Lambda^2}{4} \frac{P_4(z)}{z^4}, \quad (3.5)$$

where the quartic polynomial $P_4(z)$ is given by

$$P_4(z) = z^4 - \frac{4\mu_1}{\Lambda} z^3 + \frac{4}{\Lambda^2} \left(\mu_3^2 + c_2 - \frac{\Lambda^2}{2} \right) z^2 - \frac{4\mu_3}{\Lambda} z + 1. \quad (3.6)$$

From now on, we will take $\mu_1 = \mu_3 = m$ for simplicity. Then we have

$$P_4(z) = z^4 - \frac{4m}{\Lambda} z^3 + \frac{4A}{\Lambda^2} z^2 - \frac{4m}{\Lambda} z + 1 \quad (3.7)$$

with $A \equiv m^2 + c_2 - \Lambda^2/2$. The leading order vector multiplet vev $a_{0,0} \equiv a_0$ and the half genus contribution $a_{0,1}$ in (2.20) are written as

$$a_0 = \frac{1}{2\pi i} \oint \frac{\Lambda}{2} \frac{\sqrt{P_4(z)}}{z^2} dz, \quad (3.8)$$

$$a_{0,1} = \frac{1}{2\pi i} \oint \frac{1}{2} \left(z + \frac{1}{z} \right) \frac{dz}{\sqrt{P_4(z)}} = -\frac{1}{2} \frac{\partial a_0}{\partial m} \quad (3.9)$$

We are going to express the genus one part $a_{0,2}$ of the vev in terms of the derivatives of a_0 , which are given by

$$\frac{\partial a_0}{\partial A} = \frac{1}{2\pi i} \oint \frac{dz}{\Lambda \sqrt{P_4}}, \quad (3.10)$$

$$\frac{\partial^2 a_0}{\partial m^2} = \frac{1}{2\pi i} \oint \left(-\frac{2}{\Lambda} \right) (z^2 + 1)^2 \frac{dz}{P_4^{3/2}}, \quad (3.11)$$

$$\frac{\partial^2 a_0}{\partial m \partial A} = \frac{1}{2\pi i} \oint \frac{2}{\Lambda} (z^3 + z) \frac{dz}{P_4^{3/2}}, \quad (3.12)$$

$$\frac{\partial^2 a_0}{\partial A^2} = \frac{1}{2\pi i} \oint \left(-\frac{2}{\Lambda^3} \right) z^2 \frac{dz}{P_4^{3/2}}. \quad (3.13)$$

The $a_{0,2}$ in (2.21) consists of three terms and we will consider the three terms in turn. The first term is

$$\frac{1}{2\pi i} \oint \frac{y_0'^2}{8y_0^3} dz = \frac{1}{2\pi i} \oint \frac{1}{\Lambda} \left(\frac{z^2}{16} \frac{P_4'^2}{P_4^{5/2}} - \frac{z}{2} \frac{P_4'}{P_4^{3/2}} + \frac{1}{\sqrt{P_4}} \right) dz. \quad (3.14)$$

The first integral on the right hand side can be written as

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{z^2}{16\Lambda} \frac{P_4'^2}{P_4^{5/2}} dz &= \frac{1}{2\pi i} \oint \frac{z^2}{16\Lambda} \left(\frac{4}{3} \frac{d^2}{dz^2} \frac{1}{\sqrt{P_4}} + \frac{2}{3} \frac{P_4''}{P_4^{3/2}} \right) dz \\ &= \frac{1}{2\pi i} \oint \frac{1}{6\Lambda} \left(\frac{1}{\sqrt{P_4}} + \frac{z^2 P_4''}{4P_4^{3/2}} \right) dz, \end{aligned} \quad (3.15)$$

where we have integrated by parts in the last equality. The explicit form of $P_4(z)$ in (3.7) satisfies

$$z^2 P_4'' = 3z P_4' + \frac{12m}{\Lambda}(z^3 + z) - \frac{16A}{\Lambda^2} z^2. \quad (3.16)$$

Given the above relation and using the derivatives of a_0 in (3.10), (3.12), (3.13), we find (3.15) to be

$$\frac{1}{2\pi i} \oint \frac{z^2}{16\Lambda} \frac{P_4'^2}{P_4^{5/2}} dz = \frac{1}{6} \frac{\partial a_0}{\partial A} + \frac{1}{2\pi i} \oint \frac{z}{8\Lambda} \frac{P_4'}{P_4^{3/2}} dz + \frac{m}{4} \frac{\partial^2 a_0}{\partial m \partial A} + \frac{A}{3} \frac{\partial^2 a_0}{\partial A^2}. \quad (3.17)$$

Thus, we obtain the first term (3.14) of $a_{0,2}$ as

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{y_0'^2}{8y_0^3} dz &= \frac{7}{6} \frac{\partial a_0}{\partial A} + \frac{m}{4} \frac{\partial^2 a_0}{\partial m \partial A} + \frac{A}{3} \frac{\partial^2 a_0}{\partial A^2} + \frac{1}{2\pi i} \oint \frac{-3z}{8\Lambda} \frac{P_4'}{P_4^{3/2}} dz \\ &= \frac{5}{12} \frac{\partial a_0}{\partial A} + \frac{m}{4} \frac{\partial^2 a_0}{\partial m \partial A} + \frac{A}{3} \frac{\partial^2 a_0}{\partial A^2}, \end{aligned} \quad (3.18)$$

where we have used in the last step

$$\frac{1}{2\pi i} \oint z \frac{P_4'}{P_4^{3/2}} dz = \frac{1}{2\pi i} \oint \frac{2}{\sqrt{P_4}} dz = 2\Lambda \frac{\partial a_0}{\partial A}. \quad (3.19)$$

The second and third terms of $a_{0,2}$ in (2.21) are given by

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{(V_0'' + V_0' V_1')^2}{32y_0^3} dz &= \frac{1}{2\pi i} \oint \frac{1}{4\Lambda} (z^2 + 1)^2 \frac{dz}{P_4^{3/2}} \\ &= -\frac{1}{8} \frac{\partial^2 a_0}{\partial m^2}, \end{aligned} \quad (3.20)$$

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{V_1'^2 + 2V_1''}{8y_0} dz &= \frac{1}{2\pi i} \oint \left(-\frac{1}{4\Lambda}\right) \frac{dz}{\sqrt{P_4}} \\ &= -\frac{1}{4} \frac{\partial a_0}{\partial A}. \end{aligned} \quad (3.21)$$

Plugging (3.18), (3.20), and (3.21) into (2.21), we find

$$a_{0,2} = \frac{1}{6} \frac{\partial a_0}{\partial A} + \frac{m}{4} \frac{\partial^2 a_0}{\partial m \partial A} + \frac{A}{3} \frac{\partial^2 a_0}{\partial A^2} + \frac{1}{8} \frac{\partial^2 a_0}{\partial m^2}. \quad (3.22)$$

The planar vev (3.8) can be computed by doing elliptic integral and expanding hypergeometric function, which was done in [14].

$$\begin{aligned} a_0 = \sqrt{A} \left(1 - \frac{m^2}{4A^2} \Lambda^2 - \frac{(A^2 - 6m^2 A + 15m^4)}{64A^4} \Lambda^4 - \frac{5(3m^2 A^2 - 14m^4 A + 21m^6)}{256A^6} \Lambda^6 \right. \\ \left. - \frac{15(A^4 - 28m^2 A^3 + 294m^4 A^2 - 924m^6 A + 1001m^8)}{16384A^8} \Lambda^8 + \mathcal{O}(\Lambda^{10}) \right). \end{aligned} \quad (3.23)$$

The half genus part (3.9) is given by [15]

$$a_{0,1} = \frac{m}{4A^{3/2}}\Lambda^2 - \frac{3m(A-5m^2)}{32A^{7/2}}\Lambda^4 + \frac{5(3A^2m-28Am^3+63m^5)}{256A^{11/2}}\Lambda^6 \\ - \frac{105(m(A^3-21A^2m^2+99Am^4-143m^6))}{4096A^{15/2}}\Lambda^8 + \dots \quad (3.24)$$

The genus one part (3.22) is

$$a_{0,2} = -\frac{A+m^2}{16A^{5/2}}\Lambda^2 + \frac{A^2-30Am^2-35m^4}{128A^{9/2}}\Lambda^4 - \frac{5(3A^3-49A^2m^2+147Am^4+231m^6)}{1024A^{13/2}}\Lambda^6 \\ + \frac{35(A^4-126A^3m^2+792A^2m^4-858Am^6-2145m^8)}{16384A^{17/2}}\Lambda^8 + \dots \quad (3.25)$$

We invert the equation $a = a_0 + \epsilon a_{0,1} + \epsilon^2 a_{0,2}$ to find A in terms of a :

$$A = \left(a^2 + \frac{m^2}{2a^2}\Lambda^2 + \frac{a^4-6a^2m^2+5m^4}{32a^6}\Lambda^4 + \frac{5a^4m^2-14a^2m^4+9m^6}{64a^{10}}\Lambda^6 \right. \\ \left. + \frac{5a^8-252a^6m^2+1638a^4m^4-2860a^2m^6+1469m^8}{8192a^{14}}\Lambda^8 + \dots \right) \\ + \epsilon \left(-\frac{m}{2a^2}\Lambda^2 + \frac{3a^2m-5m^3}{16a^6}\Lambda^4 - \frac{5a^4m-28a^2m^3+27m^5}{64a^{10}}\Lambda^6 \right. \\ \left. + \frac{63a^6m-819a^4m^3+2145a^2m^5-1469m^7}{2048a^{14}}\Lambda^8 + \dots \right) \\ + \epsilon^2 \left(\frac{a^2+m^2}{8a^4}\Lambda^2 - \frac{a^4-21m^4}{64a^8}\Lambda^4 + \frac{5a^6-14a^4m^2-123a^2m^4+220m^6}{256a^{12}}\Lambda^6 \right. \\ \left. + \frac{-21a^8+738a^6m^2+3080a^4m^4-19266a^2m^6+18445m^8}{8192a^{16}}\Lambda^8 + \dots \right) \quad (3.26)$$

In order to obtain the free energy, we consider the relation

$$\Lambda \frac{\partial}{\partial \Lambda} F = g_s \sqrt{\beta} \Lambda \sum_I \left\langle \lambda_I + \frac{1}{\lambda_I} \right\rangle = g_s \sqrt{\beta} \Lambda \langle \sum_I \lambda_I \rangle + c_2. \quad (3.27)$$

The vev $\langle \sum_I \lambda_I \rangle$ can be evaluated by looking at large z behavior of the planar resolvent $W_{0,0}(z) \approx g_s \sqrt{\beta} N/z + g_s \sqrt{\beta} \langle \sum_I \lambda_I \rangle / z^2 + \dots$ in the loop equation (2.13).

$$g_s \sqrt{\beta} \Lambda \langle \sum_I \lambda_I \rangle = c_2 + (\mu_1^2 - \mu_3^2) + \dots \quad (3.28)$$

For $\mu_1 = \mu_3 = m$, we have

$$\Lambda \frac{\partial}{\partial \Lambda} F = 2c_2 = 2(A - m^2) + \Lambda^2 \quad (3.29)$$

up to g_s^2 . By integrating in Λ , we obtain the free energy $F = F_{0,0} + \epsilon F_{0,1} + \epsilon^2 F_{0,2} + \dots$ for $N_F = 2$ model. The planar part is given by

$$\begin{aligned} F_{0,0} = & 2(a^2 - m^2) \log \Lambda + \frac{a^2 + m^2}{2a^2} \Lambda^2 + \frac{a^4 - 6a^2 m^2 + 5m^4}{64a^6} \Lambda^4 \\ & + \frac{5a^4 m^2 - 14a^2 m^4 + 9m^6}{192a^{10}} \Lambda^6 \\ & + \frac{5a^8 - 252a^6 m^2 + 1638a^4 m^4 - 2860a^2 m^6 + 1469m^8}{32768a^{14}} \Lambda^8 + \dots, \end{aligned} \quad (3.30)$$

the half genus part by

$$\begin{aligned} F_{0,1} = & -\frac{m}{2a^2} \Lambda^2 + \frac{3a^2 m - 5m^3}{32a^6} \Lambda^4 - \frac{5a^4 m - 28a^2 m^3 + 27m^5}{192a^{10}} \Lambda^6 \\ & + \frac{63a^6 m - 819a^4 m^3 + 2145a^2 m^5 - 1469m^7}{8192a^{14}} \Lambda^8 + \dots, \end{aligned} \quad (3.31)$$

and the genus one part by

$$\begin{aligned} F_{0,2} = & \frac{a^2 + m^2}{8a^4} \Lambda^2 - \frac{a^4 - 21m^4}{128a^8} \Lambda^4 + \frac{5a^6 - 14a^4 m^2 - 123a^2 m^4 + 220m^6}{768a^{12}} \Lambda^6 \\ & + \frac{-21a^8 + 738a^6 m^2 + 3080a^4 m^4 - 19266a^2 m^6 + 18445m^8}{32768a^{16}} \Lambda^8 + \dots. \end{aligned} \quad (3.32)$$

The $F_{0,0}$ and $F_{0,1}$ are the same as the known results [14] and [15], and the genus one correction $F_{0,2}$ exactly matches the corresponding part (B.16) of the genus one correction computed from the Nekrasov partition function.

4 $N_F = 3$ Model

In this section, we consider the matrix model, which corresponds to $SU(2)$ gauge theory with $N_F = 3$. The action of the matrix model (2.4) is given by

$$V(z) = V_0 + \epsilon V_1 = 2\mu_3 \log z + 2m_1 \log(z - 1) - \frac{\Lambda}{z} + \epsilon \log z. \quad (4.1)$$

The function $f(z)$ reads

$$f(z) = \frac{c_1}{z} + \frac{c_2}{z-1} + \frac{c_3}{z^2}, \quad (4.2)$$

where c_1, c_2 and c_3 are

$$\begin{aligned} c_1 = & -g_s \sqrt{\beta} \sum_{I=1}^N \left\langle \frac{2\mu_3 + \epsilon}{\lambda_I} + \frac{\Lambda}{\lambda_I^2} \right\rangle, \quad c_2 = -g_s \sqrt{\beta} \sum_{I=1}^N \left\langle \frac{2m_1}{\lambda_I - 1} \right\rangle, \\ c_3 = & -g_s \sqrt{\beta} \sum_{I=1}^N \left\langle \frac{\Lambda}{\lambda_I} \right\rangle. \end{aligned} \quad (4.3)$$

The condition $\langle \sum_I V'(\lambda_I) \rangle = 0$ yields a constraint

$$c_1 + c_2 = 0. \quad (4.4)$$

From the large z limit of the planar loop equation in (2.13) with $W_{0,0}(z) \approx g_s \sqrt{\beta} N/z$ and $f(z) \approx (c_2 + c_3)/z^2$, we have another constraint

$$c_2 + c_3 = m_\infty^2 - (\mu_3 + m_1)^2. \quad (4.5)$$

Since only one parameter among c_i 's is independent, we choose c_3 to parameterize the spectral curve.

The planar spectral curve for $N_F = 3$ model is of the form

$$y_0^2 = \frac{P_4(z)}{4z^4(z-1)^2}, \quad (4.6)$$

where the polynomial $P_4(z)$ is

$$\begin{aligned} P_4(z) = & 4m_\infty^2 z^4 - 4(B - m_1 \Lambda - m_1^2 + m_\infty^2) z^3 \\ & + (4B - 4m_1 \Lambda + \Lambda^2 - 4\mu_3 \Lambda) z^2 + 2\Lambda(2\mu_3 - \Lambda) z + \Lambda^2, \end{aligned} \quad (4.7)$$

with $B = c_3 - \mu_3 \Lambda + \mu_3^2$. Henceforth, we take $\mu_3 = m$ and $m_1 = m_\infty = 0$ for simplicity. Then, $P_4(z)$ reduces to

$$P_4(z) = -4Bz^3 + (4B + \Lambda^2 - 4m\Lambda)z^2 + 2\Lambda(2m - \Lambda)z + \Lambda^2. \quad (4.8)$$

The planar Coulomb branch parameter a_0 and the half genus part $a_{0,1}$ are

$$a_0 = \frac{1}{2\pi i} \oint \frac{\sqrt{P_4}}{2z^2(z-1)} dz, \quad (4.9)$$

$$a_{0,1} = \frac{1}{2\pi i} \oint \frac{\Lambda}{2} \left(\frac{1}{z} - 1 \right) \frac{dz}{\sqrt{P_4}} = -\frac{1}{2} \frac{\partial a_0}{\partial m} + \frac{\Lambda}{2} \frac{\partial a_0}{\partial B}, \quad (4.10)$$

with the derivatives of a_0 ,

$$\frac{\partial a_0}{\partial m} = \frac{1}{2\pi i} \oint \left(-\frac{\Lambda}{z} \right) \frac{dz}{\sqrt{P_4}}, \quad \frac{\partial a_0}{\partial B} = \frac{1}{2\pi i} \oint \left(-\frac{dz}{\sqrt{P_4}} \right). \quad (4.11)$$

We will need the second derivatives of a_0 to compute the genus one part $a_{0,2}$.

$$\frac{\partial^2 a_0}{\partial m^2} = \frac{1}{2\pi i} \oint 2\Lambda^2(1-z) \frac{dz}{P_4^{3/2}} \quad (4.12)$$

$$\frac{\partial^2 a_0}{\partial m \partial B} = \frac{1}{2\pi i} \oint 2\Lambda z(1-z) \frac{dz}{P_4^{3/2}} \quad (4.13)$$

$$\frac{\partial^2 a_0}{\partial B^2} = \frac{1}{2\pi i} \oint 2z^2(1-z) \frac{dz}{P_4^{3/2}} \quad (4.14)$$

Now, we will find the expression of $a_{0,2}$ in terms of the derivatives of a_0 . The first term in (2.21) is given by

$$\frac{1}{2\pi i} \oint \frac{y_0'^2}{8y_0^3} dz = \frac{1}{2\pi i} \oint \frac{1}{16} \left((z^3 - z^2) \frac{P_4'^2}{P_4^{5/2}} - 4z(3z - 2) \frac{P_4'}{P_4^{3/2}} + \frac{4(3z - 2)^2}{z - 1} \frac{1}{\sqrt{P_4}} \right) dz. \quad (4.15)$$

The first integral on the right hand side of the above equation can be simplified using the fact

$$\frac{P_4'^2}{P_4^{5/2}} = \frac{4}{3} \frac{d^2}{dz^2} \frac{1}{\sqrt{P_4}} + \frac{2}{3} \frac{P_4''}{P_4^{3/2}}, \quad (4.16)$$

and integrating by parts.

$$\frac{1}{2\pi i} \oint \frac{1}{16} (z^3 - z^2) \frac{P_4'^2}{P_4^{5/2}} dz = \frac{1}{2\pi i} \oint \left(\frac{3z - 1}{6} \frac{1}{\sqrt{P_4}} + \frac{1}{24} (z^3 - z^2) \frac{P_4''}{P_4^{3/2}} \right) dz. \quad (4.17)$$

The numerator of the last term satisfies the following relation.

$$(z^3 - z^2)P_4'' = 2(z^2 - z)P_4' + 2(4B + \Lambda^2 - 4m\Lambda)(z^2 - z^3) + 4\Lambda(2m - \Lambda)(z - z^2) \quad (4.18)$$

Then, we obtain the integral (4.17) after integration by parts

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{1}{16} (z^3 - z^2) \frac{P_4'^2}{P_4^{5/2}} dz &= \frac{1}{2\pi i} \oint \frac{1}{6} (5z - 2) \frac{dz}{\sqrt{P_4}} \\ &+ \frac{1}{12} (2m - \Lambda) \frac{\partial^2 a_0}{\partial m \partial B} + \frac{1}{24} (4B + \Lambda^2 - 4m\Lambda) \frac{\partial^2 a_0}{\partial B^2}, \end{aligned} \quad (4.19)$$

via (4.13) and (4.14). Plugging the above expression in (4.15), one has

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{y_0'^2}{8y_0^3} dz &= \frac{1}{2\pi i} \oint \left(\frac{1}{12} \frac{z}{\sqrt{P_4}} + \frac{1}{4(z - 1)} \frac{1}{\sqrt{P_4}} \right) dz + \frac{1}{4} \frac{\partial a_0}{\partial B} \\ &+ \frac{1}{12} (2m - \Lambda) \frac{\partial^2 a_0}{\partial m \partial B} + \frac{1}{24} (4B + \Lambda^2 - 4m\Lambda) \frac{\partial^2 a_0}{\partial B^2}. \end{aligned} \quad (4.20)$$

We give the computation of the integrals in the above equation in Appendix A. With (A.7) and (A.10), we obtain the first term of $a_{0,2}$ as

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{y_0'^2}{8y_0^3} dz &= \frac{1}{12B} a_0 + \frac{1}{12} \frac{\partial a_0}{\partial B} - \frac{m}{12B} \frac{\partial a_0}{\partial m} \\ &- \frac{1}{8\Lambda} (4m - \Lambda) \frac{\partial^2 a_0}{\partial m^2} + \frac{1}{12\Lambda} (-12B + 2m\Lambda - \Lambda^2) \frac{\partial^2 a_0}{\partial m \partial B} \\ &+ \frac{1}{24} (4B + \Lambda^2 - 4m\Lambda) \frac{\partial^2 a_0}{\partial B^2}. \end{aligned} \quad (4.21)$$

The second term of $a_{0,2}$ in (2.21) takes the form

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{(V_0'' + V_0' V_1')^2}{32y_0^3} dz &= \frac{1}{2\pi i} \oint \frac{\Lambda^2}{4} ((z^3 - z^2) + 2(z - z^2) - (1 - z)) \frac{dz}{P_4^{3/2}} \\ &= -\frac{\Lambda^2}{8} \frac{\partial^2 a_0}{\partial B^2} + \frac{\Lambda}{4} \frac{\partial^2 a_0}{\partial m \partial B} - \frac{1}{8} \frac{\partial^2 a_0}{\partial m^2}. \end{aligned} \quad (4.22)$$

The third term of $a_{0,2}$ is given by

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{V_1'^2 + 2V_1''}{8y_0} &= \frac{1}{2\pi i} \oint \frac{1-z}{4\sqrt{P_4}} dz \\ &= -\frac{1}{4B} a_0 + \frac{1}{4} \frac{\partial a_0}{\partial B} + \frac{m}{4B} \frac{\partial a_0}{\partial m}, \end{aligned} \quad (4.23)$$

where we have used the result of (A.7). Collecting all three terms (4.21), (4.22), and (4.23) of $a_{0,2}$, we find that

$$\begin{aligned} a_{0,2} &= -\frac{1}{6B} a_0 + \frac{1}{3} \frac{\partial a_0}{\partial B} + \frac{m}{6B} \frac{\partial a_0}{\partial m} \\ &\quad + \frac{1}{4\Lambda} (-2m + \Lambda) \frac{\partial^2 a_0}{\partial m^2} + \frac{1}{6\Lambda} (-6B + m\Lambda - 2\Lambda^2) \frac{\partial^2 a_0}{\partial m \partial B} \\ &\quad + \frac{1}{6} (B - m\Lambda + \Lambda^2) \frac{\partial^2 a_0}{\partial B^2}. \end{aligned} \quad (4.24)$$

The planar and the half genus contribution of the Coulomb branch parameter have been computed, which we reproduce here. [14], [15]

$$\begin{aligned} a_0 &= -\sqrt{B} \left(1 + \frac{m}{4B} \Lambda - \frac{B + 3m^2}{64B^2} \Lambda^2 + \frac{m(5m^2 + B)}{256B^3} \Lambda^3 \right. \\ &\quad - \frac{3B^2 + 30m^2B + 175m^4}{16384B^4} \Lambda^4 + \frac{m(9B^2 + 70m^2B + 441m^4)}{65536B^5} \Lambda^5 \\ &\quad \left. - \frac{5B^3 + 105m^2B^2 + 735m^4B + 4851m^6}{1048576B^6} \Lambda^6 + \mathcal{O}(\Lambda^7) \right). \end{aligned} \quad (4.25)$$

$$\begin{aligned} a_{0,1} &= -\frac{1}{8\sqrt{B}} \Lambda + \frac{m}{64B^{3/2}} \Lambda^2 - \frac{B + 3m^2}{512B^{5/2}} \Lambda^3 + \frac{9Bm + 25m^3}{8192B^{7/2}} \Lambda^4 \\ &\quad - \frac{9B^2 + 90Bm^2 + 245m^4}{65536B^{9/2}} \Lambda^5 + \frac{m(75B^2 + 490Bm^2 + 1323m^4)}{524288B^{11/2}} \Lambda^6 + \dots \end{aligned} \quad (4.26)$$

The genus one part (4.24) is given by

$$\begin{aligned}
a_{0,2} = & -\frac{1}{12\sqrt{B}} + \frac{m}{12B^{3/2}}\Lambda + \frac{11B - 51m^2}{768B^{5/2}}\Lambda^2 + \frac{95m^3 - 12Bm}{1536B^{7/2}}\Lambda^3 \\
& + \frac{111B^2 + 1350Bm^2 - 11725m^4}{196608B^{9/2}}\Lambda^4 - \frac{7(15B^2m + 185Bm^3 - 1638m^5)}{196608B^{11/2}}\Lambda^5 \\
& + \frac{1315B^3 + 32235B^2m^2 + 381465Bm^4 + 635481m^6}{25165824B^{13/2}}\Lambda^6 + \dots .
\end{aligned} \tag{4.27}$$

Inverting the vev a to get B , we obtain

$$\begin{aligned}
B = & \left(a^2 - \frac{m\Lambda}{2} + \frac{a^2 + m^2}{32a^2}\Lambda^2 + \frac{a^4 - 6m^2a^2 + 5m^4}{8192a^6}\Lambda^4 + \frac{5a^4m^2 - 14a^2m^4 + 9m^6}{262144a^{10}}\Lambda^6 + \dots \right) \\
& + \epsilon \left(-\frac{\Lambda}{4} - \frac{m}{32a^2}\Lambda^2 + \frac{3a^2m - 5m^3}{4096a^6}\Lambda^4 - \frac{5a^4m - 28a^2m^3 + 27m^5}{262144a^{10}}\Lambda^6 + \dots \right) \\
& + \epsilon^2 \left(-\frac{1}{6} + \frac{m\Lambda}{8a^2} + \frac{a^2 - m^2}{64a^4}\Lambda^2 + \frac{5m^3 - 3a^2m}{256a^6}\Lambda^3 - \frac{2a^4 - 25a^2m^2 + 21m^4}{8192a^8}\Lambda^4 \right. \\
& \quad + \frac{35a^4m - 210a^2m^3 + 207m^5}{65536a^{10}}\Lambda^5 \\
& \quad \left. + \frac{215a^6 + 5719a^4m^2 + 75477a^2m^4 + 691933m^6}{4194304a^{12}}\Lambda^6 + \dots \right) .
\end{aligned} \tag{4.28}$$

The free energy can be computed from the derivative of F with respect to Λ .

$$\Lambda \frac{\partial}{\partial \Lambda} F = -g_s \sqrt{\beta} \sum_I \left\langle \frac{\Lambda}{\lambda_I} \right\rangle = c_3 = B + m\Lambda - m^2 . \tag{4.29}$$

The planar and the half genus part of the free energy are integrated to

$$\begin{aligned}
F_{0,0} = & (a^2 - m^2) \log \Lambda + \frac{m\Lambda}{2} + \frac{a^2 + m^2}{64a^2}\Lambda^2 \\
& + \frac{a^4 - 6m^2a^2 + 5m^4}{32768a^6}\Lambda^4 + \frac{5a^4m^2 - 14a^2m^4 + 9m^6}{1572864a^{10}}\Lambda^6 + \dots .
\end{aligned} \tag{4.30}$$

$$F_{0,1} = -\frac{\Lambda}{4} - \frac{m}{64a^2}\Lambda^2 + \frac{3a^2m - 5m^3}{16384a^6}\Lambda^4 - \frac{5a^4m - 28a^2m^3 + 27m^5}{1572864a^{10}}\Lambda^6 + \dots . \tag{4.31}$$

The genus one part is given by

$$\begin{aligned}
F_{0,2} = & -\frac{1}{6} \log \Lambda + \frac{m}{8a^2} \Lambda + \frac{a^2 - m^2}{128a^4} \Lambda^2 + \frac{5m^3 - 3a^2m}{768a^6} \Lambda^3 - \frac{2a^4 - 25a^2m^2 + 21m^4}{32768a^8} \Lambda^4 \\
& + \frac{35a^4m - 210a^2m^3 + 207m^5}{327680a^{10}} \Lambda^5 \\
& + \frac{215a^6 + 5719a^4m^2 + 75477a^2m^4 + 691933m^6}{25165824a^{12}} \Lambda^6 + \dots .
\end{aligned} \tag{4.32}$$

Again, the free energy of the matrix model agrees with that of the gauge theory computed from the Nekrasov partition function in Appendix B.2.

5 $N_F = 4$ Model

In this section, we consider the matrix model with action (2.5), which is related to the superconformal case of $SU(2)$ gauge theory with four flavors. The Penner type potential reads

$$\begin{aligned}
V_0(z) &= 2m_0 \log z + 2m_1 \log(z-1) + 2m_2 \log(z-q), \\
V_1(z) &= \log z.
\end{aligned} \tag{5.1}$$

We evaluate the function $f(z)$ in (2.12) as

$$f(z) = \frac{c_0}{z} + \frac{c_1}{z-1} + \frac{c_2}{z-q}, \tag{5.2}$$

where the c_i 's are given by

$$\begin{aligned}
c_0 &= -g_s \sqrt{\beta} \sum_I \left\langle \frac{2m_0 + \epsilon}{\lambda_I} \right\rangle, & c_1 &= -g_s \sqrt{\beta} \sum_I \left\langle \frac{2m_1}{\lambda_I - 1} \right\rangle, \\
c_2 &= -g_s \sqrt{\beta} \sum_I \left\langle \frac{2m_2}{\lambda_I - q} \right\rangle.
\end{aligned} \tag{5.3}$$

From the equation of motion $\langle \sum_I V'(\lambda_I) \rangle = 0$, it follows that

$$\sum_{i=0}^2 c_i = 0. \tag{5.4}$$

And from the asymptotic behavior for $z \rightarrow \infty$ of the leading order loop equation in (2.13), one finds that the parameters satisfy

$$c_1 + qc_2 = m_\infty^2 - \left(\sum_{i=0}^2 m_i \right)^2. \tag{5.5}$$

Therefore, we have a single parameter left to describe the spectral curve, which we take to be c_0 .

The planar spectral curve is

$$y_0^2 = \frac{P_4(z)}{z^2(z-1)^2(z-q)^2}, \quad (5.6)$$

with $P_4(z)$ a polynomial of degree four. We will consider the case where the mass μ_i of all four hypermultiplets is equal to m such that the mass parameters m_i are set to $m_0 = m_\infty = 0$ and $m_1 = m_2 = m$. Then, the polynomial $P_4(z)$ becomes

$$P_4(z) = Cz^3 + ((1-q)^2m^2 - C(1+q))z^2 + Cqz, \quad (5.7)$$

with $C \equiv qc_0$.

In this case, the Coulomb branch parameters up to half genus are

$$a_0 = \frac{1}{2\pi i} \oint \frac{1}{z(z-1)(z-q)} \sqrt{P_4} dz, \quad (5.8)$$

$$\begin{aligned} a_{0,1} &= \frac{1}{2\pi i} \oint \left(-\frac{m(1+q)}{2} \frac{1}{\sqrt{P_4}} - \frac{m(1-q)^2}{2} \frac{z}{(z-1)(z-q)} \frac{dz}{\sqrt{P_4}} \right) \\ &= -m(1+q) \frac{\partial a_0}{\partial C} - \frac{1}{2} \frac{\partial a_0}{\partial m}, \end{aligned} \quad (5.9)$$

where we have utilized in the last equality

$$\frac{\partial a_0}{\partial C} = \frac{1}{2\pi i} \oint \frac{1}{2} \frac{dz}{\sqrt{P_4}}, \quad (5.10)$$

$$\frac{\partial a_0}{\partial m} = \frac{1}{2\pi i} \oint m(1-q)^2 \frac{z}{(z-1)(z-q)} \frac{dz}{\sqrt{P_4}}. \quad (5.11)$$

We compute the second derivatives of a_0 as

$$\frac{\partial^2 a_0}{\partial C \partial m} = \frac{1}{2\pi i} \oint \frac{-m(1-q)^2}{2} z^2 \frac{dz}{P_4^{3/2}}, \quad (5.12)$$

$$\frac{\partial^2 a_0}{\partial m^2} = \frac{1}{2\pi i} \oint \left(\frac{-m^2(1-q)^4 z^3}{(z-1)(z-q)} \frac{1}{P_4^{3/2}} + \frac{(1-q)^2 z}{(z-1)(z-q)} \frac{1}{\sqrt{P_4}} \right) dz. \quad (5.13)$$

Now, we calculate the genus one part $a_{0,2}$ in terms of the derivatives of a_0 . The first term of $a_{0,2}$ in (2.21) becomes

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{y_0'^2}{8y_0^3} dz &= \frac{1}{2\pi i} \oint \left(\frac{z(z-1)(z-q)}{32} \frac{P_4'^2}{P_4^{5/2}} - \frac{3z^2 - 2(1+q)z + q}{8} \frac{P_4'}{P_4^{3/2}} \right. \\ &\quad \left. + \frac{(3z^2 - 2(1+q)z + q)^2}{8z(z-1)(z-q)} \frac{1}{\sqrt{P_4}} \right) dz. \end{aligned} \quad (5.14)$$

Using (4.16) and integrating by parts, the first integral on the right hand side can be written as

$$\frac{1}{2\pi i} \oint \left(\frac{3z - (1+q)}{12} \frac{1}{\sqrt{P_4}} + \frac{z(z-1)(z-q)}{48} \frac{P_4''}{P_4^{3/2}} \right) dz. \quad (5.15)$$

This can be written using (A.14) as

$$\begin{aligned} & \frac{1}{2\pi i} \oint \left(\frac{3}{8} \frac{z}{\sqrt{P_4}} + \frac{m^2(1-q)^2 q}{8} \frac{z}{P_4^{3/2}} \right) dz - \frac{1}{4}(1+q) \frac{\partial a_0}{\partial C} - \frac{m^2(1-q)^2}{6C} \frac{\partial a_0}{\partial C} \\ & - \frac{m(m^2(1-q)^2 - C(1+q))}{6C} \frac{\partial^2 a_0}{\partial C \partial m}. \end{aligned} \quad (5.16)$$

The second term in (5.14) is integrated by parts to

$$\frac{1}{2\pi i} \oint \left(-\frac{3}{2}z + \frac{1}{2}(1+q) \right) \frac{dz}{\sqrt{P_4}} = \frac{1}{2\pi i} \oint \frac{-3}{2} \frac{z}{\sqrt{P_4}} dz + (1+q) \frac{\partial a_0}{\partial C}. \quad (5.17)$$

The third term in (5.14) can be written as

$$\begin{aligned} & \frac{1}{2\pi i} \oint \frac{1}{8} \left(9z - 3(1+q) + \frac{q}{z} + (1-q)^2 \frac{z}{(z-1)(z-q)} \right) \frac{dz}{\sqrt{P_4}} \\ & = \frac{1}{2\pi i} \oint \frac{9}{8} \frac{z}{\sqrt{P_4}} dz - \frac{1}{2\pi i} \oint \frac{m^2(1-q)^2 - C(1+q)}{8} \frac{qz}{P_4^{3/2}} dz \\ & - \frac{3}{4}(1+q) \frac{\partial a_0}{\partial C} + \frac{Cq}{2m(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m} + \frac{1}{8m} \frac{\partial a_0}{\partial m}, \end{aligned} \quad (5.18)$$

where (A.21) has been used. Adding (5.16), (5.17), and (5.18), the first term of $a_{0,2}$ in (5.14) becomes

$$\begin{aligned} & \frac{1}{2\pi i} \oint \frac{y_0'^2}{8y_0^3} dz = \frac{1}{2\pi i} \oint \frac{Cq(1+q)}{8} \frac{z}{P_4^{3/2}} dz \\ & - \frac{m^2(1-q)^2}{6C} \frac{\partial a_0}{\partial C} - \frac{m(m^2(1-q)^2 - C(1+q))}{6C} \frac{\partial^2 a_0}{\partial C \partial m} \\ & + \frac{Cq}{2m(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m} + \frac{1}{8m} \frac{\partial a_0}{\partial m}. \end{aligned} \quad (5.19)$$

For the integration of $z/P_4^{3/2}$ in the above equation, we use (A.17), which results in

$$\begin{aligned} & \frac{1}{2\pi i} \oint \frac{y_0'^2}{8y_0^3} dz = \frac{1}{8}(1+q) \frac{\partial a_0}{\partial C} + \frac{1+q}{m(1-q)^2} (m^2(1-q)^2 - C(1+q)) \frac{\partial^2 a_0}{\partial C \partial m} \\ & - \frac{m^2(1-q)^2}{6C} \frac{\partial a_0}{\partial C} - \frac{m(m^2(1-q)^2 - C(1+q))}{6C} \frac{\partial^2 a_0}{\partial C \partial m} \\ & + \frac{Cq}{2m(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m} + \frac{1}{8m} \frac{\partial a_0}{\partial m}. \end{aligned} \quad (5.20)$$

Next, we compute the second term of $a_{0,2}$ in (2.21).

$$\begin{aligned}
& \frac{1}{2\pi i} \oint \frac{(V_0'' + V_0' V_1')^2}{32y_0^3} dz \\
&= \frac{1}{2\pi i} \oint \frac{m^2}{8} \left(\frac{(1-q)^4 z^3}{(z-1)(z-q)} + (1+q)^2 z^3 + (1+q)^3 z^2 - 8q(1+q)z^2 + q(1+q)^2 z \right) \frac{dz}{P_4^{3/2}} \\
&= -\frac{1}{8} \frac{\partial^2 a_0}{\partial m^2} + \frac{1}{8m} \frac{\partial a_0}{\partial m} \\
&\quad + \frac{1}{2\pi i} \oint \frac{m^2}{8} \left(\frac{(1+q)^2}{C} P_4 - \frac{(1+q)^2}{C} (m^2(1-q)^2 - 2C(1+q))z^2 - 8q(1+q)z^2 \right) \frac{dz}{P_4^{3/2}} \\
&= -\frac{1}{8} \frac{\partial^2 a_0}{\partial m^2} + \frac{1}{8m} \frac{\partial a_0}{\partial m} + \frac{m^2(1+q)^2}{4C} \frac{\partial a_0}{\partial C} \\
&\quad + \frac{m}{4C} \frac{(1+q)^2}{(1-q)^2} (m^2(1-q)^2 - 2C(1+q)) \frac{\partial^2 a_0}{\partial C \partial m} + 2m \frac{q(1+q)}{(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m}. \tag{5.21}
\end{aligned}$$

The third term of the genus one part of the Coulomb branch parameter (2.21) is calculated to

$$\begin{aligned}
& \frac{1}{2\pi i} \oint \frac{V_1'^2 + 2V_1''}{8y_0} dz \\
&= \frac{1}{2\pi i} \oint \frac{-1}{8} \left(z - (1+q) + \frac{q}{z} \right) \frac{dz}{\sqrt{P_4}} \\
&= \frac{m^2(1-q)^2 - C(1+q)}{8C} \frac{\partial a_0}{\partial C} + \frac{(m^2(1-q)^2 - C(1+q))^2}{8Cm(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m} \\
&\quad - \frac{Cq}{2m(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m} + \frac{1}{4}(1+q) \frac{\partial a_0}{\partial C} - \frac{Cq}{2m(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m} \\
&\quad + \frac{1}{2\pi i} \oint \frac{q}{8} (m^2(1-q)^2 - C(1+q)) \frac{z}{P_4^{3/2}} dz \\
&= \frac{m^2(1-q)^2}{4C} \frac{\partial a_0}{\partial C} + \frac{(m^2(1-q)^2 - C(1+q))^2}{4Cm(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m} \\
&\quad - \frac{Cq}{m(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m}, \tag{5.22}
\end{aligned}$$

where (A.20), (5.10), and (A.21) have been used in the second equality and (A.17) was used in the last equality. From the results of the three terms of $a_{0,2}$ in (5.20), (5.21), and

(5.22), we find the genus one part of the Coulomb branch parameter (2.21)

$$\begin{aligned}
a_{0,2} = & \frac{1}{8}(1+q)\frac{\partial a_0}{\partial C} + \frac{m^2(1-q)^2}{12C}\frac{\partial a_0}{\partial C} - \frac{Cq}{2m(1-q)^2}\frac{\partial^2 a_0}{\partial C\partial m} \\
& + \frac{1}{8m}\frac{1+q}{(1-q)^2}(m^2(1-q)^2 - C(1+q))\frac{\partial^2 a_0}{\partial C\partial m} - \frac{m}{6C}(m^2(1-q)^2 - C(1+q))\frac{\partial^2 a_0}{\partial C\partial m} \\
& + \frac{1}{8}\frac{\partial^2 a_0}{\partial m^2} - \frac{m^2(1+q)^2}{4C}\frac{\partial a_0}{\partial C} - \frac{m}{4C}\frac{(1+q)^2}{(1-q)^2}(m^2(1-q)^2 - C(1+q))\frac{\partial^2 a_0}{\partial C\partial m} \\
& + \frac{m}{4}\frac{(1+q)^3}{(1-q)^2}\frac{\partial^2 a_0}{\partial C\partial m} - 2m\frac{q(1+q)}{(1-q)^2}\frac{\partial^2 a_0}{\partial C\partial m} \\
& + \frac{(m^2(1-q)^2 - C(1+q))^2}{4Cm(1-q)^2}\frac{\partial^2 a_0}{\partial C\partial m}. \tag{5.23}
\end{aligned}$$

We give a few leading order terms in m of the planar and the half genus contribution of the Coulomb branch parameter. [14], [15]

$$a_0 = i\sqrt{C} \left(h_0(q) - h_1(q)\frac{m^2}{C} - \frac{h_2(q)}{3}\frac{m^4}{C^2} - \frac{h_3(q)}{5}\frac{m^6}{C^3} - \frac{h_4(q)}{7}\frac{m^8}{C^4} + \mathcal{O}\left(\frac{m^{10}}{C^5}\right) \right),$$

$$a_{0,1} = i \left(g_1(q)\frac{m}{\sqrt{C}} + g_2(q)\frac{m^3}{C^{3/2}} + g_3(q)\frac{m^5}{C^{5/2}} + g_4(q)\frac{m^7}{C^{7/2}} + \dots \right), \tag{5.24}$$

where the functions $h_i(q)$'s are obtained from the expansion of the hypergeometric function,

$$h_0(q) = 1 + \frac{1}{4}q + \frac{9}{64}q^2 + \frac{25}{256}q^3 + \frac{1225}{16384}q^4 + \mathcal{O}(q^5), \tag{5.25}$$

$$h_1(q) = \frac{1}{2} + \frac{1}{8}q + \frac{1}{128}q^2 + \frac{1}{512}q^3 + \frac{25}{32768}q^4 + \mathcal{O}(q^5), \tag{5.26}$$

$$h_2(q) = \frac{3}{8} + \frac{27}{32}q + \frac{27}{512}q^2 + \frac{3}{2048}q^3 + \frac{27}{131072}q^4 + \mathcal{O}(q^5), \tag{5.27}$$

$$h_3(q) = \frac{5}{16} + \frac{125}{64}q + \frac{1125}{1024}q^2 + \frac{125}{4096}q^3 + \frac{125}{262144}q^4 + \mathcal{O}(q^5), \tag{5.28}$$

$$h_4(q) = \frac{35}{128} + \frac{1715}{512}q + \frac{42875}{8192}q^2 + \frac{42875}{32768}q^3 + \frac{42875}{2097152}q^4 + \mathcal{O}(q^5) \tag{5.29}$$

and the function $g_i(q)$ is given in terms of $h_i(q)$'s by

$$g_1(q) = \frac{2h_1(q) - (1+q)h_0(q)}{2}, \tag{5.30}$$

$$g_2(q) = \frac{4h_2(q) - 3(1+q)h_1(q)}{6}, \tag{5.31}$$

$$g_3(q) = \frac{6h_3(q) - 5(1+q)h_2(q)}{10}, \tag{5.32}$$

$$g_4(q) = \frac{8h_4(q) - 7(1+q)h_3(q)}{14}. \tag{5.33}$$

The genus one part of the vev (5.23) is computed to

$$a_{0,2} = i \left(k_1(q) \frac{1}{\sqrt{C}} + k_2(q) \frac{m^2}{C^{3/2}} + k_3(q) \frac{m^4}{C^{5/2}} + k_4(q) \frac{m^6}{C^{7/2}} + \dots \right), \quad (5.34)$$

where $k_i(q)$ is related to $h_i(q)$'s as

$$k_1(q) = -\frac{1}{16}(- (q+1)h_0(q) + 2h_1(q)), \quad (5.35)$$

$$k_2(q) = -\frac{1}{48} (4(q^2 + 4q + 1)h_0(q) - 17(q+1)h_1(q) + 12h_2(q)), \quad (5.36)$$

$$k_3(q) = -\frac{1}{48} (12(q^2 + 4q + 1)h_1(q) - 31(q+1)h_2(q) + 18h_3(q)), \quad (5.37)$$

$$k_4(q) = -\frac{1}{48} (20(q^2 + 4q + 1)h_2(q) - 45(q+1)h_3(q) + 24h_4(q)). \quad (5.38)$$

With the explicit results of the Coulomb branch parameter, we solve the equation $a = a_0 + \epsilon a_{0,1} + \epsilon^2 a_{0,2}$ for C .

$$\begin{aligned} C = & -a^2 \left(\frac{1}{h_0(q)^2} - \frac{2h_1(q)}{h_0(q)} \frac{m^2}{a^2} + \frac{2h_0(q)h_2(q) - 3h_1(q)^2}{3} \frac{m^4}{a^4} \right. \\ & \left. - \frac{10h_0(q)h_1(q)^3 - 10h_0(q)^2h_1(q)h_2(q) + 2h_0(q)^3h_3(q)}{5} \frac{m^6}{a^6} + \dots \right) \\ & + \epsilon m \left(\frac{2(1+q)h_0(q) - 4h_1(q)}{h_0(q)} + \frac{8h_0(q)h_2(q) - 12h_1(q)^2}{3} \frac{m^2}{a^2} \right. \\ & \left. - \frac{12h_0(q)\{5h_1(q)^3 - 5h_0(q)h_1(q)h_2(q) + h_0(q)^2h_3(q)\}}{5} \frac{m^4}{a^4} + \dots \right) \\ & + \epsilon^2 \left(-\frac{(1+q)h_0(q) - 2h_1(q)}{8h_0(q)} \right. \\ & + \frac{15h_1(q)^2 + h_0(q)^2(-1+q)^2 - h_0(q)(6h_2(q) + 5h_1(q)(1+q))}{12} \frac{m^2}{a^2} \\ & + \frac{h_0(q)}{12} \{81h_1(q)^3 - 3h_0(q)h_1(q)(23h_2(q) + 5(1+q)h_1(q)) \\ & \left. + h_0(q)^2(9h_3(q) + 10(1+q)h_2(q))\} \frac{m^4}{a^4} + \dots \right) \end{aligned} \quad (5.39)$$

Now, we can compute the free energy by

$$\frac{\partial}{\partial q} F = -g_s \sqrt{\beta} \left\langle \sum_I \frac{2m}{\lambda_I - q} \right\rangle = c_2. \quad (5.40)$$

From (5.5), c_2 is related to $C \equiv qc_0$, thus we obtain

$$\frac{\partial}{\partial q} F = \frac{1}{1-q} \left(4m^2 - \frac{C}{q} \right). \quad (5.41)$$

The $F_{0,0}$ and $F_{0,1}$ are given by

$$\begin{aligned}
F_{0,0} = & (a^2 - m^2) \log q + \frac{a^4 + 6a^2m^2 + m^4}{2a^2}q \\
& + \frac{(13a^8 + 100a^6m^2 + 22a^4m^4 - 12a^2m^6 + 5m^8)}{64a^6}q^2 \\
& + \frac{23a^{12} + 204a^{10}m^2 + 51a^8m^4 - 48a^6m^6 + 45a^4m^8 - 28a^2m^{10} + 9m^{12}}{192a^{10}}q^3 \\
& + \frac{1}{32768a^{14}} (2701a^{16} + 26440a^{14}m^2 + 7164a^{12}m^4 - 9000a^{10}m^6 \\
& + 12190a^8m^8 - 13384a^6m^{10} + 10908a^4m^{12} - 5720a^2m^{14} + 1469m^{16}) q^4 + \dots, \tag{5.42}
\end{aligned}$$

$$\begin{aligned}
F_{0,1} = & -\frac{2m(a^2 + m^2)}{2a^2}q - \frac{9a^6m + 11a^4m^3 - 9a^2m^5 + 5m^7}{16a^6}q^2 \\
& - \frac{38a^{10}m + 51a^8m^3 - 72a^6m^5 + 90a^4m^7 - 70a^2m^9 + 27m^{11}}{96a^{10}}q^3 \\
& - \frac{1}{4096a^{14}} (1257a^{14}m + 1791a^{12}m^3 - 3375a^{10}m^5 + 6095a^8m^7 - 8365a^6m^9 \\
& + 8181a^4m^{11} - 5005a^2m^{13} + 1469m^{15}) q^4 + \dots. \tag{5.43}
\end{aligned}$$

Finally, we obtain the genus one free energy $F_{0,2}$.

$$\begin{aligned}
F_{0,2} = & \frac{a^4 + 6a^2m^2 + m^4}{8a^4}q + \frac{9a^8 + 64a^6m^2 - 70a^4m^4 + 40a^2m^6 + 21m^8}{128a^8}q^2 \\
& + \frac{19a^{12} + 147a^{10}m^2 - 300a^8m^4 + 470a^6m^6 - 357a^4m^8 + 39a^2m^{10} + 110m^{12}}{384a^{12}}q^3 \\
& + \frac{1}{32768a^{16}} (1257a^{16} + 10276a^{14}m^2 - 28776a^{12}m^4 + 67460a^{10}m^6 - 105630a^8m^8 \\
& + 103308a^6m^{10} - 43120a^4m^{12} - 15028a^2m^{14} + 18445m^{16}) q^4 + \dots. \tag{5.44}
\end{aligned}$$

The genus one part is identical to (B.6), which is calculated in $\mathcal{N} = 2$ $SU(2)$ superconformal gauge theory with four flavors.

6 Summary and Discussion

In this paper, we have investigated the β -deformed matrix models with logarithmic potentials, which have been suggested to explain the AGT relation. Specifically, we have computed the genus one part of the free energy in the matrix models, which describe $\mathcal{N} = 2$ $SU(2)$ gauge theories with $N_F = 2, 3$, and 4 flavors. We have checked that the results obtained from the matrix model nicely match the free energies computed from the Nekrasov partition function.

It would be interesting to generalize the duality to $SU(N_c)$ gauge group, in which case a multi-matrix model should be considered. [16] [17] One can also consider extending the results to higher genus parts of the free energy. [33], [34], [35] The β -deformed matrix model is also related to the refinement of the topological B-model. [36] The study of matrix model along this line would be highly interesting.

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A Calculation of Integrals

In this appendix, we show the details of calculations of the results used throughout the paper.

A.1 Integrals for $N_F = 3$ Model

Consider the a_0 in (4.9) in the form of

$$a_0 = \frac{1}{2\pi i} \oint \frac{1}{2} \left(\frac{1}{z-1} - \frac{1}{z} - \frac{1}{z^2} \right) \sqrt{P_4} dz. \quad (\text{A.1})$$

We will rewrite the three terms in the above equation in turn. Through the relation

$$\frac{1}{z-1} = (-4Bz^2 + (-4m\Lambda + \Lambda^2)z - \Lambda^2) \frac{1}{P_4}, \quad (\text{A.2})$$

the first integral is written as

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{1}{z-1} \sqrt{P_4} dz &= \frac{1}{2\pi i} \oint (-4Bz^2 + (-4m\Lambda + \Lambda^2)z - \Lambda^2) \frac{dz}{\sqrt{P_4}} \\ &= \frac{1}{2\pi i} \oint \left(-\frac{1}{3}(8B + 4m\Lambda - \Lambda^2)z - \frac{1}{3}(4m\Lambda + \Lambda^2) \right) \frac{dz}{\sqrt{P_4}}, \quad (\text{A.3}) \end{aligned}$$

where the following condition has been used to eliminate z^2 term.

$$\begin{aligned}
0 &= \frac{1}{2\pi i} \oint \frac{d}{dz} \sqrt{P_4} dz = \frac{1}{2\pi i} \oint \frac{P'_4}{2\sqrt{P_4}} dz \\
&= \frac{1}{2\pi i} \oint (-6Bz^2 + (4B - 4m\Lambda + \Lambda^2)z + \Lambda(2m - \Lambda)) \frac{dz}{\sqrt{P_4}} \quad (\text{A.4})
\end{aligned}$$

Similar manipulations give the second term in (A.1).

$$\begin{aligned}
\frac{1}{2\pi i} \oint \frac{1}{z} \sqrt{P_4} dz &= \frac{1}{2\pi i} \oint \left(-4Bz^2 + (4B - 4m\Lambda + \Lambda^2)z + 2\Lambda(2m - \Lambda) + \frac{\Lambda^2}{z} \right) \frac{dz}{\sqrt{P_4}} \\
&= \frac{1}{2\pi i} \oint \left(\frac{1}{3}(4B - 4m\Lambda + \Lambda^2) + \frac{4}{3}\Lambda(2m - \Lambda) + \frac{\Lambda^2}{z} \right) \frac{dz}{\sqrt{P_4}} \quad (\text{A.5})
\end{aligned}$$

The third term of (A.1) can be written as

$$\begin{aligned}
\frac{1}{2\pi i} \oint \frac{1}{z^2} \sqrt{P_4} dz &= \frac{1}{2\pi i} \oint \frac{1}{z} \left(\frac{P'_4}{2\sqrt{P_4}} \right) dz \\
&= \frac{1}{2\pi i} \oint \left(-6Bz + (4B - 4m\Lambda + \Lambda^2) + \frac{\Lambda(2m - \Lambda)}{z} \right) \frac{dz}{\sqrt{P_4}}. \quad (\text{A.6})
\end{aligned}$$

Adding (A.3), (A.5), and (A.6), one finds (A.1) to be

$$a_0 = \frac{1}{2\pi i} \oint Bz \frac{dz}{\sqrt{P_4}} + 2B \frac{\partial a_0}{\partial B} + m \frac{\partial a_0}{\partial m}, \quad (\text{A.7})$$

where (4.11) has been used.

Next, we rewrite the following integral

$$\begin{aligned}
\frac{1}{2\pi i} \oint \frac{1}{z-1} \frac{dz}{\sqrt{P_4}} &= \frac{1}{2\pi i} \oint (-4Bz^2 + (-4m\Lambda + \Lambda^2)z - \Lambda^2) \frac{dz}{P_4^{3/2}} \\
&= \frac{2B}{\Lambda} \frac{\partial^2 a_0}{\partial m \partial B} + \frac{1}{2\pi i} \oint (- (4B + 4m\Lambda - \Lambda^2)z - \Lambda^2) \frac{dz}{P_4^{3/2}} \\
&= \frac{2B}{\Lambda} \frac{\partial^2 a_0}{\partial m \partial B} + \frac{1}{2\Lambda^2} (4B + 4m\Lambda - \Lambda^2) \frac{\partial^2 a_0}{\partial m^2} \\
&\quad + \frac{1}{2\pi i} \oint (-4B - 4m\Lambda) \frac{dz}{P_4^{3/2}}, \quad (\text{A.8})
\end{aligned}$$

where (4.13) and (4.12) have been used in the second and third equality. The integral in

the last line can be obtained from the relation

$$\begin{aligned}
0 &= \frac{1}{2\pi i} \oint \frac{P_4'}{P_4^{3/2}} dz \\
&= \frac{1}{2\pi i} \oint (-12Bz^2 + 2(4B - 4m\Lambda + \Lambda^2)z + 4m\Lambda - 2\Lambda^2) \frac{dz}{P_4^{3/2}} \\
&= \frac{6B}{\Lambda} \frac{\partial^2 a_0}{\partial m \partial B} + \frac{1}{\Lambda^2} (2B + 4m\Lambda - \Lambda^2) \frac{\partial^2 a_0}{\partial m^2} \\
&\quad + \frac{1}{2\pi i} \oint (-4B - 4m\Lambda) \frac{dz}{P_4^{3/2}}. \tag{A.9}
\end{aligned}$$

Thus, we find the integral (A.8) to be

$$\frac{1}{2\pi i} \oint \frac{1}{z-1} \frac{dz}{\sqrt{P_4}} = -\frac{4B}{\Lambda} \frac{\partial^2 a_0}{\partial m \partial B} + \frac{1}{2\Lambda^2} (-4m\Lambda + \Lambda^2) \frac{\partial^2 a_0}{\partial m^2}. \tag{A.10}$$

A.2 Integrals for $N_F = 4$ Model

First, we consider the integral in (5.15)

$$\begin{aligned}
&\frac{1}{2\pi i} \oint z(z-1)(z-q) \frac{P_4''}{P_4^{3/2}} dz \\
&= \frac{1}{2\pi i} \oint \frac{1}{C} (P_4 - m^2(1-q)^2 z^2) \frac{P_4''}{P_4^{3/2}} dz \\
&= \frac{1}{2\pi i} \oint \left(\frac{6Cz - 2C(1+q) + 2m^2(1-q)^2}{C\sqrt{P_4}} - \frac{m^2(1-q)^2}{C} \frac{z^2 P_4''}{P_4^{3/2}} \right) dz. \tag{A.11}
\end{aligned}$$

We rewrite the second term in the last line via

$$z^2 P_4'' = 6P_4 - 4(m^2(1-q)^2 - C(1+q))z^2 - 6Cqz, \tag{A.12}$$

and (5.12) to obtain

$$\begin{aligned}
\frac{1}{2\pi i} \oint \frac{-m^2(1-q)^2}{C} \frac{z^2 P_4''}{P_4^{3/2}} dz &= \frac{1}{2\pi i} \oint \left(\frac{-6m^2(1-q)^2}{C} \frac{1}{\sqrt{P_4}} + 6m^2(1-q)^2 q \frac{z}{P_4^{3/2}} \right) dz \\
&\quad - \frac{8m}{C} (m^2(1-q)^2 - C(1+q)) \frac{\partial^2 a_0}{\partial C \partial m}. \tag{A.13}
\end{aligned}$$

Substituting the above equation into (A.11), we get

$$\begin{aligned}
& \frac{1}{2\pi i} \oint z(z-1)(z-q) \frac{P_4''}{P_4^{3/2}} dz \\
&= \frac{1}{2\pi i} \oint \left(6z - 2(1+q) - \frac{4m^2(1-q)^2}{C} \right) \frac{dz}{\sqrt{P_4}} + \frac{1}{2\pi i} \oint 6m^2(1-q)^2 q \frac{z}{P_4^{3/2}} dz \\
&\quad - \frac{8m}{C} (m^2(1-q)^2 - C(1+q)) \frac{\partial^2 a_0}{\partial C \partial m} \\
&= \frac{1}{2\pi i} \oint \left(\frac{6z}{\sqrt{P_4}} + 6m^2(1-q)^2 q \frac{z}{P_4^{3/2}} \right) dz - \left(4(1+q) + \frac{8m^2(1-q)^2}{C} \right) \frac{\partial a_0}{\partial C} \\
&\quad - \frac{8m}{C} (m^2(1-q)^2 - C(1+q)) \frac{\partial^2 a_0}{\partial C \partial m}, \tag{A.14}
\end{aligned}$$

where (5.10) has been used in the last equality. In the above expression, we need to compute the integrals of $z/\sqrt{P_4}$ and $z/P_4^{3/2}$, which we do in the following.

So, we rewrite $\partial a_0/\partial C$ of (5.10) to obtain the integral of $z/P_4^{3/2}$ through

$$zP_4' = 3P_4 - (m^2(1-q)^2 - C(1+q))z^2 - 2Cqz, \tag{A.15}$$

such that

$$\begin{aligned}
\frac{\partial a_0}{\partial C} &= \frac{1}{2\pi i} \oint \frac{dz}{2\sqrt{P_4}} = \frac{1}{2\pi i} \oint \frac{z}{4} \frac{P_4'}{P_4^{3/2}} dz \\
&= \frac{1}{2\pi i} \oint \frac{1}{4} (3P_4 - (m^2(1-q)^2 - C(1+q))z^2 - 2Cqz) \frac{dz}{P_4^{3/2}} \\
&= \frac{3}{2} \frac{\partial a_0}{\partial C} + \frac{m^2(1-q)^2 - C(1+q)}{2m(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m} + \frac{1}{2\pi i} \oint \frac{-Cqz}{2} \frac{dz}{P_4^{3/2}}. \tag{A.16}
\end{aligned}$$

Therefore, we obtain

$$\frac{1}{2\pi i} \oint \frac{Cqz}{P_4^{3/2}} dz = \frac{\partial a_0}{\partial C} + \frac{m^2(1-q)^2 - C(1+q)}{m(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m}. \tag{A.17}$$

Next, we will compute the integral of $z/\sqrt{P_4}$ using the relation

$$\begin{aligned}
z^2 P_4' &= 3zP_4 - \frac{m^2(1-q)^2 - C(1+q)}{C} P_4 - 2Cqz^2 \\
&\quad + \frac{(m^2(1-q)^2 - C(1+q))^2}{C} z^2 + (m^2(1-q)^2 - C(1+q))qz. \tag{A.18}
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{1}{2\pi i} \oint \frac{z}{\sqrt{P_4}} dz &= \frac{1}{2\pi i} \oint \frac{z^2}{4} \frac{P'_4}{P_4^{3/2}} \\
&= \frac{1}{2\pi i} \oint \left(\frac{3z}{4\sqrt{P_4}} - \frac{m^2(1-q)^2 - C(1+q)}{4C} \frac{1}{\sqrt{P_4}} - \frac{Cqz^2}{2P_4^{3/2}} \right. \\
&\quad \left. + \frac{(m^2(1-q)^2 - C(1+q))^2}{4C} \frac{z^2}{P_4^{3/2}} + \frac{m^2(1-q)^2 - C(1+q)}{4} \frac{qz}{P_4^{3/2}} \right) dz \\
&= \frac{1}{2\pi i} \oint \frac{3}{4} \frac{z}{\sqrt{P_4}} dz - \frac{m^2(1-q)^2 - C(1+q)}{2C} \frac{\partial a_0}{\partial C} + \frac{Cq}{m(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m} \\
&\quad - \frac{(m^2(1-q)^2 - C(1+q))^2}{2Cm(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m} \\
&\quad + \frac{m^2(1-q)^2 - C(1+q)}{4C} \left(\frac{\partial a_0}{\partial C} + \frac{m^2(1-q)^2 - C(1+q)}{m(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m} \right), \tag{A.19}
\end{aligned}$$

where (A.17) has been used in the last line. Thus, we obtain

$$\begin{aligned}
\frac{1}{2\pi i} \oint \frac{z}{\sqrt{P_4}} dz &= -\frac{m^2(1-q)^2 - C(1+q)}{C} \frac{\partial a_0}{\partial C} \\
&\quad - \frac{(m(1-q)^2 - C(1+q))^2}{Cm(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m} + \frac{4Cq}{m(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m}. \tag{A.20}
\end{aligned}$$

The integral used in (5.18) is computed as

$$\begin{aligned}
\frac{1}{2\pi i} \oint \frac{1}{z} \frac{dz}{\sqrt{P_4}} &= \frac{1}{2\pi i} \oint (Cz^2 + (m^2(1-q)^2 - C(1+q))z + Cq) \frac{dz}{P_4^{3/2}} \\
&= \frac{1}{2\pi i} \oint (-2Cz^2 - (m^2(1-q)^2 - C(1+q))z) \frac{dz}{P_4^{3/2}} \\
&= \frac{4C}{m(1-q)^2} \frac{\partial^2 a_0}{\partial C \partial m} - \frac{1}{2\pi i} \oint (m^2(1-q)^2 - C(1+q))z \frac{dz}{P_4^{3/2}}, \tag{A.21}
\end{aligned}$$

where in the second equality the relation

$$\begin{aligned}
0 &= \frac{1}{2\pi i} \oint \frac{P'_4}{P_4^{3/2}} dz \\
&= \frac{1}{2\pi i} \oint (3Cz^2 + 2(m^2(1-q)^2 - C(1+q))z + Cq) \frac{dz}{P_4^{3/2}} \tag{A.22}
\end{aligned}$$

was used to replace the constant term.

B Nekrasov Partition Function

We give a brief summary of the Nekrasov instanton partition function and present relevant results obtained from them. [37], [38]

B.1 $N_F = 4$ theory

The instanton part of the Nekrasov partition function for $\mathcal{N} = 2$ $U(2)$ gauge theory with four anti-fundamental hypermultiplets is of the form

$$Z_{\text{inst}} = \sum_{\vec{Y}} q^{|\vec{Y}|} Z_{\text{vec}}(\vec{a}, \vec{Y}) \prod_{i=1}^4 Z_{\text{afund}}(\vec{a}, \vec{Y}, \mu_i), \quad (\text{B.1})$$

where $\vec{Y} = (Y_1, Y_2)$ is a pair of Young diagrams, $\vec{a} = (a_1, a_2)$ a pair of Coulomb branch parameters and μ_i denotes the mass of the hypermultiplet. The vector multiplet and anti-fundamental hypermultiplet contributions are given by

$$Z_{\text{vec}}(\vec{a}, \vec{Y}) = \prod_{i,j=1}^2 \prod_{s \in Y_i} (a_{ij} - \epsilon_1 L_{Y_j}(s) + \epsilon_2 (A_{Y_i}(s) + 1))^{-1} \prod_{t \in Y_j} (a_{ji} + \epsilon_1 (L_{Y_j}(t) + 1) - \epsilon_2 A_{Y_i}(t))^{-1},$$

$$Z_{\text{afund}}(\vec{a}, \vec{Y}, \mu) = \prod_{i=1}^2 \prod_{s \in Y_i} (a_i + \epsilon_1 (m - 1) + \epsilon_2 (n - 1) + \mu), \quad (\text{B.2})$$

where $a_{ij} \equiv a_i - a_j$, and the leg-length $L_{Y_i}(s) = \lambda'_n - m$ and the arm-length $A_{Y_i}(s) = \lambda_m - n$ are defined for a box s at position (m, n) in a Young diagram $Y_i = (\lambda_1 \geq \lambda_2 \geq \dots)$, with its transpose $Y_i^t = (\lambda'_1 \geq \lambda'_2 \geq \dots)$.

To compare with the results from matrix models, we consider $SU(2)$ gauge group with the Coulomb branch parameter $\vec{a} = (a, -a)$, noting that the contributions coming from the $U(1)$ factors are irrelevant for genus one correction. We define the free energy as

$$\begin{aligned} \mathcal{F} &\equiv -\epsilon_1 \epsilon_2 \log Z_{\text{inst}} \\ &= \sum_{k,l=0} g_s^{2k} \epsilon^l \mathcal{F}_{k,l}. \end{aligned} \quad (\text{B.3})$$

For the case of $\mu_i = m$, which we considered in the matrix model computation, we calculate

the free energy up to g_s^2 order.

$$\begin{aligned}
\mathcal{F}_{0,0} = & \frac{a^4 + 6a^2m^2 + m^4}{2a^2}q + \frac{13a^8 + 100a^6m^2 + 22a^4m^4 - 12a^2m^6 + 5m^8}{64a^6}q^2 \\
& + \frac{23a^{12} + 204a^{10}m^2 + 51a^8m^4 - 48a^6m^6 + 45a^4m^8 - 28a^2m^{10} + 9m^{12}}{192a^{10}}q^3 \\
& + \frac{1}{32768a^{14}} (2701a^{16} + 26440a^{14}m^2 + 7164a^{12}m^4 - 9000a^{10}m^6 \\
& + 12190a^8m^8 - 13384a^6m^{10} + 10908a^4m^{12} - 5720a^2m^{14} + 1469m^{16}) q^4 \\
& + \mathcal{O}(q^5), \tag{B.4}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{0,1} = & -\frac{m(a^2 + m^2)}{a^2}q - \frac{9a^6m + 11a^4m^3 - 9a^2m^5 + 5m^7}{16a^6}q^2 \\
& - \frac{38a^{10}m + 51a^8m^3 - 72a^6m^5 + 90a^4m^7 - 70a^2m^9 + 27m^{11}}{96a^{10}}q^3 \\
& - \frac{1}{4096a^{14}} (1257a^{14}m + 1791a^{12}m^3 - 3375a^{10}m^5 + 6095a^8m^7 - 8365a^6m^9 \\
& + 8181a^4m^{11} - 5005a^2m^{13} + 1469m^{15}) q^4 + \mathcal{O}(q^5), \tag{B.5}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{0,2} = & \frac{a^4 + 6a^2m^2 + m^4}{8a^4}q + \frac{9a^8 + 64a^6m^2 - 70a^4m^4 + 40a^2m^6 + 21m^8}{128a^8}q^2 \\
& + \frac{19a^{12} + 147a^{10}m^2 - 300a^8m^4 + 470a^6m^6 - 357a^4m^8 + 39a^2m^{10} + 110m^{12}}{384a^{12}}q^3 \\
& + \frac{1}{32768a^{16}} (1257a^{16} + 10276a^{14}m^2 - 28776a^{12}m^4 + 67460a^{10}m^6 - 105630a^8m^8 \\
& + 103308a^6m^{10} - 43120a^4m^{12} - 15028a^2m^{14} + 18445m^{16}) q^4 + \mathcal{O}(q^5), \tag{B.6}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{1,0} = & \frac{m^2(-a^2 + m^2)^3}{32a^8}q^2 + \frac{m^2(-a^2 + m^2)^3(3a^4 - 3a^2m^2 + 8m^4)}{96a^{12}}q^3 \\
& + \frac{m^2(-a^2 + m^2)^3(235a^8 - 432a^6m^2 + 1486a^4m^4 - 1656a^2m^6 + 1647m^8)}{8192a^{16}}q^4 + \mathcal{O}(q^5). \tag{B.7}
\end{aligned}$$

B.2 $N_F = 3$ theory

The instanton partition function for this case is

$$Z_{\text{inst}} = \sum_{\vec{Y}} \Lambda_3^{|\vec{Y}|} Z_{\text{vec}}(\vec{a}, \vec{Y}) \prod_{i=1}^3 Z_{\text{afund}}(\vec{a}, \vec{Y}, \mu_i), \quad (\text{B.8})$$

where Λ_3 is the dynamical scale of the gauge theory. As in section 4, we set $\mu_1 = \mu_2 = 0$ and $\mu_3 = m$. The free energy is computed as

$$\mathcal{F}_{0,0} = \frac{m}{2} \Lambda + \frac{a^2 + m^2}{64a^2} \Lambda^2 + \frac{a^4 - 6a^2m^2 + 5m^4}{32768a^6} \Lambda^4 + \dots, \quad (\text{B.9})$$

$$\mathcal{F}_{0,1} = -\frac{\Lambda}{4} - \frac{m}{64a^2} \Lambda^2 + \frac{m(3a^2 - 5m^2)}{16384a^6} \Lambda^4 + \dots, \quad (\text{B.10})$$

$$\begin{aligned} \mathcal{F}_{0,2} = & \frac{m}{8a^2} \Lambda + \frac{a^2 - m^2}{128a^4} \Lambda^2 + \frac{m(-3a^2 + 5m^2)}{768a^6} \Lambda^3 \\ & - \frac{2a^4 - 25a^2m^2 + 21m^4}{32768a^8} \Lambda^4 + \dots, \end{aligned} \quad (\text{B.11})$$

$$\mathcal{F}_{1,0} = -\frac{a^2 - 2m^2}{128a^4} \Lambda^2 + \frac{3a^4 - 41a^2m^2 + 46m^4}{32768a^8} \Lambda^4 + \dots. \quad (\text{B.12})$$

B.3 $N_F = 2$ theory

In this case, the instanton partition function is of the form

$$Z_{\text{inst}} = \sum_{\vec{Y}} \Lambda_2^{2|\vec{Y}|} Z_{\text{vec}}(\vec{a}, \vec{Y}) \prod_{i=1,3} Z_{\text{afund}}(\vec{a}, \vec{Y}, \mu_i). \quad (\text{B.13})$$

By setting $\mu_1 = \mu_3 = m$ as in section 3, we find the free energy as

$$\begin{aligned} \mathcal{F}_{0,0} = & \frac{a^2 + m^2}{2a^2} \Lambda^2 + \frac{a^4 - 6a^2m^2 + 5m^4}{64a^6} \Lambda^4 + \frac{5a^4m^2 - 14a^2m^4 + 9m^6}{192a^{10}} \Lambda^6 \\ & + \frac{5a^8 - 252a^6m^2 + 1638a^4m^4 - 2860a^2m^6 + 1469m^8}{32768a^{14}} \Lambda^8 + \dots, \end{aligned} \quad (\text{B.14})$$

$$\begin{aligned}\mathcal{F}_{0,1} = & -\frac{m}{2a^2}\Lambda^2 + \frac{m(3a^2 - 5m^2)}{32a^6}\Lambda^4 - \frac{m(5a^4 - 28a^2m^2 + 27m^4)}{192a^{10}}\Lambda^6 \\ & + \frac{m(63a^6 - 819a^4m^2 + 2145a^2m^4 - 1469m^6)}{8192a^{14}}\Lambda^8 + \dots, \quad (\text{B.15})\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{0,2} = & \frac{a^2 + m^2}{8a^4}\Lambda^2 - \frac{a^4 - 21m^4}{128a^8}\Lambda^4 + \frac{5a^6 - 14a^4m^2 - 123a^2m^4 + 220m^6}{768a^{12}}\Lambda^6 \\ & + \frac{-21a^8 + 738a^6m^2 + 3080a^4m^4 - 19266a^2m^6 + 18445m^8}{32768a^{16}}\Lambda^8 + \dots, \quad (\text{B.16})\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{1,0} = & \frac{a^4 - 3a^2m^2 + 2m^4}{64a^8}\Lambda^4 + \frac{11a^4m^2 - 27a^2m^4 + 16m^6}{192a^{12}}\Lambda^6 \\ & + \frac{23a^8 - 615a^6m^2 + 3895a^4m^4 - 6597a^2m^6 + 3294m^8}{16384a^{16}}\Lambda^8 + \dots. \quad (\text{B.17})\end{aligned}$$

References

- [1] D. Gaiotto, “N=2 dualities,” *JHEP* **1208**, 034 (2012) [arXiv:0904.2715 [hep-th]].
- [2] L. F. Alday, D. Gaiotto and Y. Tachikawa, “Liouville Correlation Functions from Four-dimensional Gauge Theories,” *Lett. Math. Phys.* **91**, 167 (2010) [arXiv:0906.3219 [hep-th]].
- [3] N. Wyllard, “A(N-1) conformal Toda field theory correlation functions from conformal N = 2 SU(N) quiver gauge theories,” *JHEP* **0911**, 002 (2009) [arXiv:0907.2189 [hep-th]].
- [4] A. Mironov and A. Morozov, “On AGT relation in the case of U(3),” *Nucl. Phys. B* **825**, 1 (2010) [arXiv:0908.2569 [hep-th]].
- [5] D. Gaiotto, “Asymptotically free N=2 theories and irregular conformal blocks,” arXiv:0908.0307 [hep-th].
- [6] A. Marshakov, A. Mironov and A. Morozov, “On non-conformal limit of the AGT relations,” *Phys. Lett. B* **682**, 125 (2009) [arXiv:0909.2052 [hep-th]].
- [7] R. Dijkgraaf and C. Vafa, “Toda Theories, Matrix Models, Topological Strings, and N=2 Gauge Systems,” arXiv:0909.2453 [hep-th].
- [8] N. Seiberg and E. Witten, “Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory,” *Nucl. Phys. B* **426**, 19 (1994) [Erratum-ibid. *B* **430**, 485 (1994)] [hep-th/9407087].
- [9] N. Seiberg and E. Witten, “Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD,” *Nucl. Phys. B* **431**, 484 (1994) [hep-th/9408099].
- [10] A. Marshakov, A. Mironov and A. Morozov, “Generalized matrix models as conformal field theories: Discrete case,” *Phys. Lett. B* **265**, 99 (1991).

- [11] S. Kharchev, Marshakov, A., A. Mironov, A. Morozov and S. Pakuliak, “Conformal matrix models as an alternative to conventional multimatrix models,” Nucl. Phys. B **404**, 717 (1993) [hep-th/9208044].
- [12] H. Itoyama and N. Yonezawa, “ ϵ -Corrected Seiberg-Witten Prepotential Obtained From Half Genus Expansion in beta-Deformed Matrix Model,” Int. J. Mod. Phys. A **26**, 3439 (2011) [arXiv:1104.2738 [hep-th]].
- [13] T. Eguchi and K. Maruyoshi, “Penner Type Matrix Model and Seiberg-Witten Theory,” JHEP **1002**, 022 (2010) [arXiv:0911.4797 [hep-th]].
- [14] T. Eguchi and K. Maruyoshi, “Seiberg-Witten theory, matrix model and AGT relation,” JHEP **1007**, 081 (2010) [arXiv:1006.0828 [hep-th]].
- [15] T. Nishinaka and C. Rim, “ β -Deformed Matrix Model and Nekrasov Partition Function,” JHEP **1202**, 114 (2012) [arXiv:1112.3545 [hep-th]].
- [16] H. Itoyama, K. Maruyoshi and T. Oota, “The Quiver Matrix Model and 2d-4d Conformal Connection,” Prog. Theor. Phys. **123**, 957 (2010) [arXiv:0911.4244 [hep-th]].
- [17] R. Schiappa and N. Wyllard, “An A(r) threesome: Matrix models, 2d CFTs and 4d N=2 gauge theories,” J. Math. Phys. **51**, 082304 (2010) [arXiv:0911.5337 [hep-th]].
- [18] A. Mironov, A. Morozov and S. Shakirov, “Matrix Model Conjecture for Exact BS Periods and Nekrasov Functions,” JHEP **1002** (2010) 030 [arXiv:0911.5721 [hep-th]].
- [19] M. Fujita, Y. Hatsuda and T. -S. Tai, “Genus-one correction to asymptotically free Seiberg-Witten prepotential from Dijkgraaf-Vafa matrix model,” JHEP **1003** (2010) 046 [arXiv:0912.2988 [hep-th]].
- [20] P. Sulkowski, “Matrix models for beta-ensembles from Nekrasov partition functions,” JHEP **1004** (2010) 063 [arXiv:0912.5476 [hep-th]].
- [21] H. Itoyama and T. Oota, “Method of Generating q-Expansion Coefficients for Conformal Block and N=2 Nekrasov Function by beta-Deformed Matrix Model,” Nucl. Phys. B **838**, 298 (2010) [arXiv:1003.2929 [hep-th]].
- [22] A. Morozov and S. Shakirov, “The matrix model version of AGT conjecture and CIV-DV prepotential,” JHEP **1008** (2010) 066 [arXiv:1004.2917 [hep-th]].
- [23] H. Itoyama, T. Oota and N. Yonezawa, “Massive Scaling Limit of beta-Deformed Matrix Model of Selberg Type,” Phys. Rev. D **82**, 085031 (2010) [arXiv:1008.1861 [hep-th]].
- [24] K. Maruyoshi and F. Yagi, “Seiberg-Witten curve via generalized matrix model,” JHEP **1101** (2011) 042 [arXiv:1009.5553 [hep-th]].
- [25] G. Bonelli, K. Maruyoshi, A. Tanzini and F. Yagi, “Generalized matrix models and AGT correspondence at all genera,” JHEP **1107** (2011) 055 [arXiv:1011.5417 [hep-th]].
- [26] H. Itoyama and T. Oota, “An(1) Affine Quiver Matrix Model,” Nucl. Phys. B **852** (2011) 336 [arXiv:1106.1539 [hep-th]].
- [27] D. Galakhov, A. Mironov and A. Morozov, “S-duality as a beta-deformed Fourier transform,” JHEP **1208**, 067 (2012) [arXiv:1205.4998 [hep-th]].
- [28] J. -E. Bourgin, “Large N limit of beta-ensembles and deformed Seiberg-Witten relations,” JHEP **1208**, 046 (2012) [arXiv:1206.1696 [hep-th]].
- [29] D. Kreff, “Penner Type Ensemble for Gauge Theories Revisited,” Phys. Rev. D **87**, 045027 (2013) [arXiv:1209.6009 [hep-th]].

- [30] J. -E. Bourguine, “Large N techniques for Nekrasov partition functions and AGT conjecture,” arXiv:1212.4972 [hep-th].
- [31] M. Billo, M. Frau, L. Gallot, A. Lerda and I. Pesando, “Deformed N=2 theories, generalized recursion relations and S-duality,” arXiv:1302.0686 [hep-th].
- [32] L. Chekhov, “Logarithmic potential β -ensembles and Feynman graphs,” arXiv:1009.5940 [math-ph].
- [33] L. O. Chekhov, B. Eynard and O. Marchal, “Topological expansion of β -ensemble model and quantum algebraic geometry in the sectorwise approach,” Theor. Math. Phys. **166**, 141 (2011) [arXiv:1009.6007 [math-ph]].
- [34] L. Chekhov and B. Eynard, “Matrix eigenvalue model: Feynman graph technique for all genera,” JHEP **0612**, 026 (2006) [math-ph/0604014].
- [35] A. Brini, M. Marino and S. Stevan, “The Uses of the refined matrix model recursion,” J. Math. Phys. **52**, 052305 (2011) [arXiv:1010.1210 [hep-th]].
- [36] M. Aganagic, M. C. N. Cheng, R. Dijkgraaf, D. Krefl and C. Vafa, “Quantum Geometry of Refined Topological Strings,” JHEP **1211**, 019 (2012) [arXiv:1105.0630 [hep-th]].
- [37] N. A. Nekrasov, “Seiberg-Witten prepotential from instanton counting,” Adv. Theor. Math. Phys. **7**, 831 (2004) [hep-th/0206161].
- [38] N. Nekrasov and A. Okounkov, “Seiberg-Witten theory and random partitions,” hep-th/0306238.